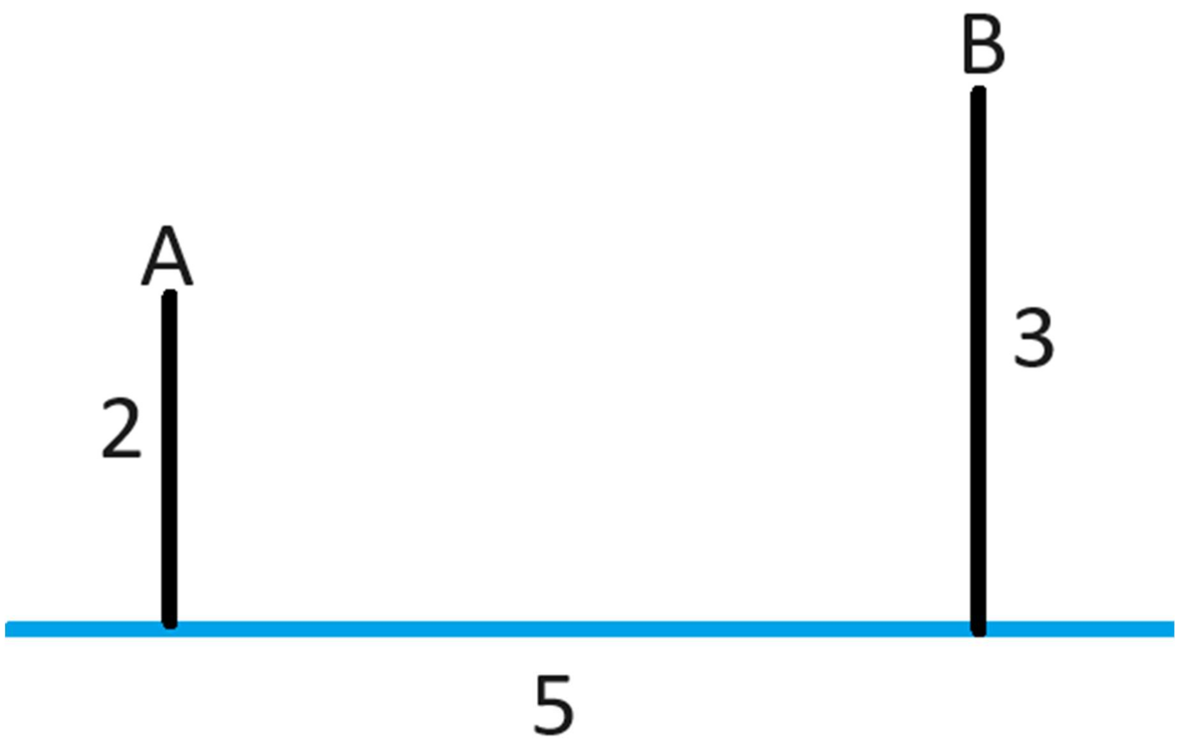
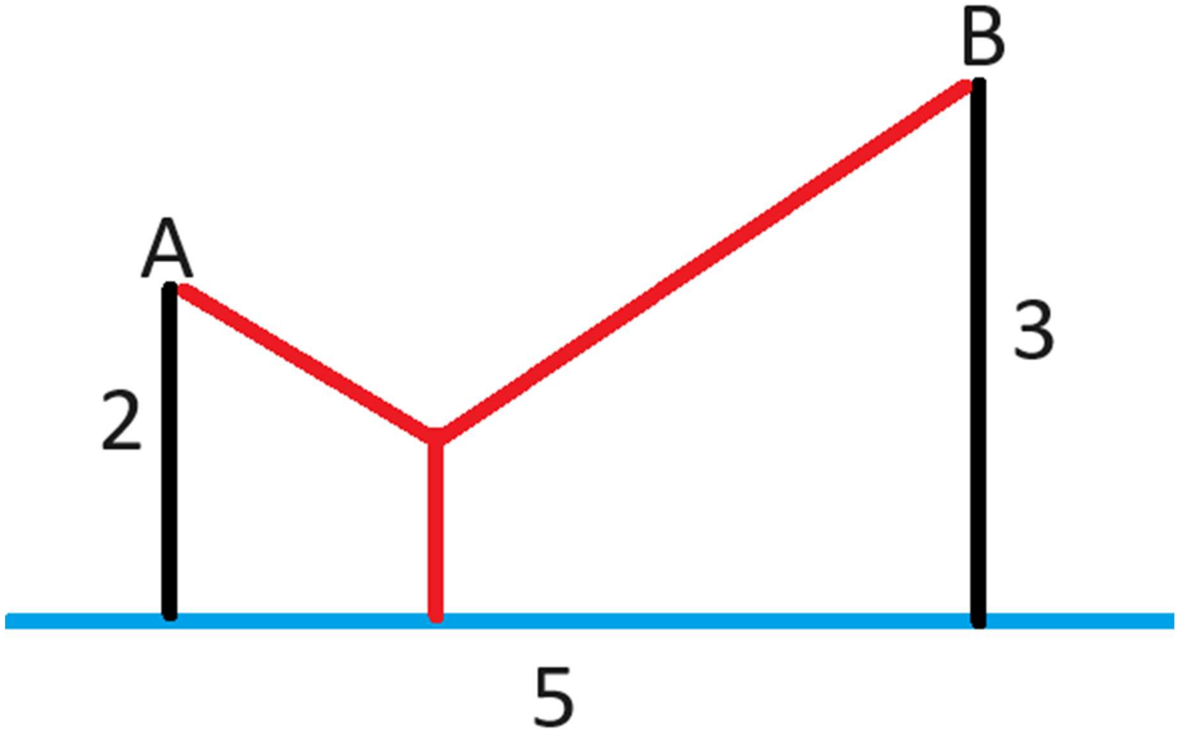
Question

There is a straight water pipe (blue) in the vicinity of points A and B. Point A is 2 miles from the closest point on the pipe. Point B is 3 miles from the closest point on the pipe. These two points along the pipe that mark the closest points to A and B are 5 miles apart. It is desired to provide water to points A and B by laying new pipes linking points A and B to anywhere along the blue pipe. These new pipes may be in any form you wish. What is the least distance of new pipe needed?



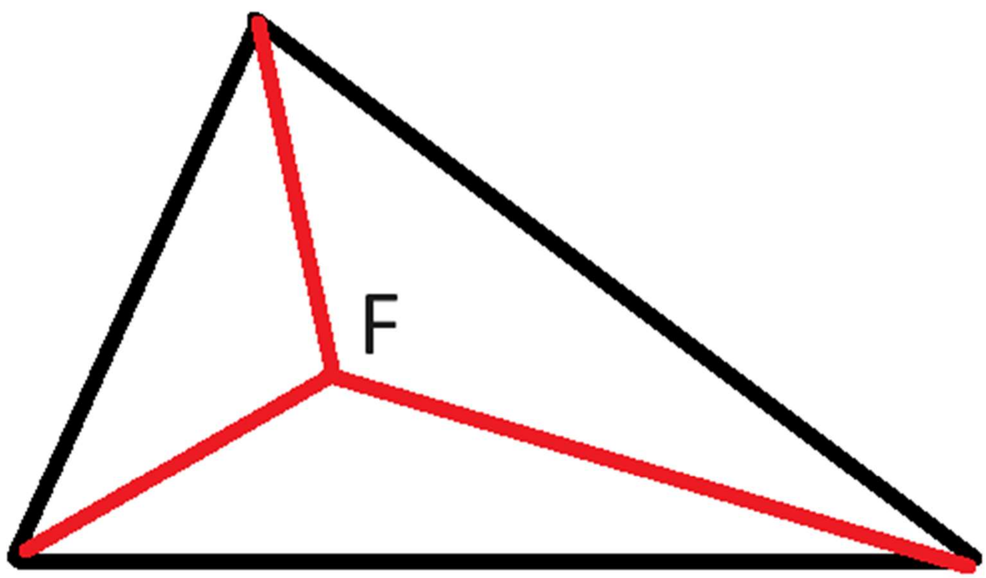
Hint #1



The new pipes should form a Y shape, as shown in the image above.

Hint #2

Consider any triangle. The task is to find the point where the sum of the three distances from this point to each corner of the triangle is a minimum.



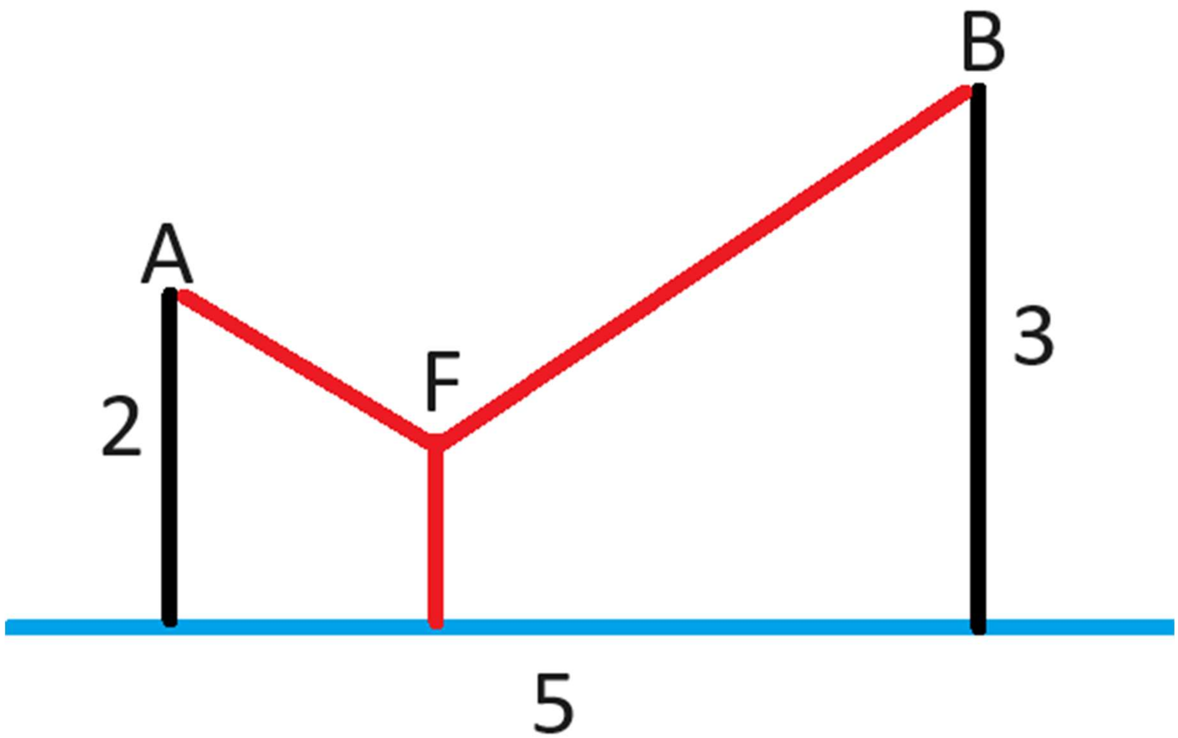
Fermat showed how to locate this point. He noted that the three angles created by this point extending to the three corners of the triangle were always 120 degrees.

For more information, visit en.wikipedia.org/wiki/Fermat_point .

Answer

Approximately 6.83012701892219 miles

Solution



Consider the diagram above, where F is the Fermat point that minimizes the sum of the distances to A, B and the closest point on the blue pipe.

As noted in hint #2, the three angles made with the ride pipes are all 120 degrees.

Simple geometry shows us the sides of a 30-60-90 triangle are proportional to 1, sqrt(3) and 2.

Let the point on the pipe closest to point A be at coordinates (0,0).

Consider the line that contains points A and F. The slope of that line is $-\frac{\sqrt{3}}{3}$. The y intercept is 2. The equation of that line is:

$$y = -\frac{\sqrt{3}}{3}\,x + 2$$

Consider the line that contains points B and F.

Recall that the general formula for the slope of the line containing points (a,b) and (c,d) is (b-d)/(a-c).

We also know the slope of the line containing B and F is $+\frac{\sqrt{3}}{3}$.

Let the y intercept of the line containing B and F be q.

Setting the known slope to the general formula we get:

$$+\frac{\sqrt{3}}{3} = \frac{3-q}{5}$$

$$5\sqrt{3} = 9 - 3q$$

$$3q = 9 - 5\sqrt{3}$$

$$q = \frac{9-5\sqrt{3}}{3}$$

Thus, the equation of the line containing B and F is:

$$y = +\frac{\sqrt{3}}{3}\,x + \frac{9-5\sqrt{3}}{3}$$

To find where the lines meet, equate the two equations for y and solve for x:

$$-\frac{\sqrt{3}}{3}\,x + 2 = +\frac{\sqrt{3}}{3}\,x + \frac{9-5\sqrt{3}}{3}$$

$$2\,\frac{\sqrt{3}}{3}\,x = 2 - \frac{9\ - 5\sqrt{3}}{3}$$

$$x = \frac{3}{2\sqrt{3}}\,(2 - \frac{9 - 5\sqrt{3}}{3})$$

$$x = (\frac{6}{2\sqrt{3}} - \frac{27 - 1\ \ \sqrt{3}}{6\sqrt{3}})$$

$$x = (\frac{18}{6\sqrt{3}} - \frac{27 - 15\sqrt{3}}{6\sqrt{3}})$$

$$x = \frac{15\sqrt{3} - 9}{6\sqrt{3}}$$

$$x = \frac{5\sqrt{3} - 3}{2\sqrt{3}}$$

$$x = \frac{15 - 3\sqrt{3}}{6}$$

$$x = \frac{5 - \sqrt{3}}{2} = \sim 1.633975$$

Recall:

$$y = -\frac{\sqrt{3}}{3}\,x + 2$$

Putting the value above for x into that equation gives us:

$$y = \frac{15 - 5\sqrt{3}}{6} =\sim 1.056624$$

We're almost done! Let's use the Pythagorean formula to find the distance of each pipe segment:

A to F = 1.886751

B to F = 3.886751

River to F = 1.056624

The sum of these distances is 6.830127.