Question

$1^2 + 2^2 + 3^2 + \ldots + n^2 =$

Answer

$1^2 + 2^2 + 3^2 + \ldots + n^2 = n\ (n+1)\ (2n+1)/6$

Solution

To get to the answer we first must find the formula for 1+2+3+…+n. Let's start by working that out.

If n is even:

1+2+3+…+n =

(1+n) + (2+(n-1)) + (3+(n-2)) + …

All these terms sum to n+1 and there are n/2 such terms. Thus, the sum is (n/2)*(n+1) = $\frac{n(n+1)}{2}$

If n is odd:

Let's omit the last term for now, leaving us with an even number. We can use the logic above, taking (n-1)/2 pairs of sums of n+1 to get a total of: (n-1)*n/2. =

$(n^2 - n)/2$. Now let's add that last term of n:

$(n^2 - n)/2 + n =$

$(n^2 - n)/2 + 2n/2 =$

$(n^2 + n)/2 =$

$$\frac{n(n+1)}{2}$$

So, the answer is n(n+1)/2 whether n is odd or even.

Now we're ready to move onto $1^2 + 2^2 + 3^2 + \ldots + n^2$

The humdinger with this solution is to use telescoping sums. Note that:

$x^3 - (x - 1)^3 =$

$x^3 - (x^3 - 3x^2 + 3x - 1) =$

$3x^2 - 3x + 1$

Next consider:

$x^3 - (x - 1)^3 = 3x^2 - 3x + 1$

$(x - 1)^3 - (x - 2)^3 = 3(x - 1)^2 - 3(x - 1) + 1$

$(x - 2)^3 - (x - 3)^3 = 3(x - 2)^2 - 3(x - 2) + 1$

$(x - 3)^3 - (x - 4)^3 = 3(x - 3)^2 - 3(x - 3) + 1$

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$(2)^3 - (1)^3 = 3(2)^2 - 3(2) + 1$

$(1)^3 - (0)^3 = 3(1)^2 - 3(1) + 1$

Next, add up the x terms. Note all but two cancel out on the left side:

$$x^3 - 0^3 = 3\sum_{i=1}^{x} i^2 - 3\sum_{i=1}^{x} i + x$$

Let's isolate $\sum_{i=1}^{x} i^2$ on the left side, since that what's we're trying to solve for.

$$3\sum_{i=1}^{x} i^2 = x^3 + 3\sum_{i=1}^{x} i - x$$

Recall, we solved for $\sum_{i=1}^{x} i$ at the beginning of this solution as x(x+1)/2.

$$3\sum_{i=1}^{x} i^2 = x^3 + 3\,x\,(x+1)/2 - x$$

$$6\sum_{i=1}^{x} i^2 = 2x^3 + 3\,x\,(x+1) - 2x$$

$$6\sum_{i=1}^{x} i^2 = 2x^3 + 3x^2 + 3x - 2x$$

$$6\sum_{i=1}^{x} i^2 = 2x^3 + 3x^2 + x$$

$$6\sum_{i=1}^{x} i^2 = x(2x^2 + 3\,x + 1)$$

$$6\sum_{i=1}^{x} i^2 = x\,(2x+1)\,(x+1)$$

$$\sum_{i=1}^{x} i^2 = \frac{x\,(2x+1)\,(x+1)}{6}$$

Since the original question had n as the last term, let's substitute that for x.

$$1^2 + 2^2 + 3^2 + \ldots + n^2 = \sum_{i=1}^{n} i^2 = \frac{n\,(2n+1)\,(n+1)}{6}$$