Q: A six-sided die is rolled until either of the following events happen:

A) Any side has appeared six times.
B) Every side has appeared at least once.

What is the probability event A occurs first?

Solution:

The answer will be the same if the length of time between rolls follows a memoryless (or exponential) distribution with a mean of 1.

Let's find the probability B happens first and then subtract that from one to get the probability A happens first.

B will occur if the last holdout number occurs while the other five faces have all occurred 1 to 5 times.

Let t represent the amount of time since the experiment started, where the average time between rolls is 1. The average number of times any given face should have appeared in t units of time is t/6.

Per the Poisson distribution, the following is the probability any given face has appeared 0 to 5 times after in t units of time.

- 0 times: exp(-t/6)
- 1 time: exp(-t/6) * (t/6)
- 2 times: exp(-t/6) * (t/6)^2/2!
- 3 times: exp(-t/6) * (t/6)^3/3!

- 4 times: exp(-t/6) * (t/6)^4/4!
- 5 times: exp(-t/6) * (t/6)^5/5!

The probability any given face has appeared 1 to five times in t units of time is:

exp(-t/6)*((t/6) + (t/6)^2/2! + (t/6)^3/3! + (t/6)^4/4! + (t/6)^5/5!)

The probability five difference faces have appeared one to five times each is the above taken to the fifth power, or:

(exp(-t/6)*((t/6) + (t/6)^2/2! + (t/6)^3/3! + (t/6)^4/4! + (t/6)^5/5!))^5

The probability the last holdout has not appeared is exp(-t/6).

The probability any specific number has not appeared and the other five have appeared one to five times each in t units of time is:

exp(-t/6) * (exp(-t/6)*((t/6) + (t/6)^2/2! + (t/6)^3/3! + (t/6)^4/4! + (t/6)^5/5!))^5

To get the probability of any face being the last holdout, multiply by 6:

6 * exp(-t/6) * (exp(-t/6)*((t/6) + (t/6)^2/2! + (t/6)^3/3! + (t/6)^4/4! + (t/6)^5/5!))^5

Let’s simplify that:

6 * (exp(-t/6))^6 * ((t/6) + (t/6)^2/2! + (t/6)^3/3! + (t/6)^4/4! + (t/6)^5/5!))^5

= 6 * exp(-t) * (t/6 + t^2/72 + t^3/ 1296 + t^4/31104 + t^5/933120)^5

The probability of the last holdout number being rolled at any given moment is the above times 1/6:

(1/6) * 6 * exp(-t) * (t/6 + t^2/72 + t^3/ 1296 + t^4/31104 + t^5/933120)^5 =

exp(-t) * (t/6 + t^2/72 + t^3/ 1296 + t^4/31104 + t^5/933120)^5 =

To get the probability of event B occurring, integrate the above over t from 0 to infinity.

$$\left(\frac{t^5}{993120}+\frac{t^4}{31104}+\frac{t^3}{1296}+\frac{t^2}{72}+\frac{t}{6}\right)^5 \mathrm{e}^{-t}$$

Using an integral calculator, like the one at https://www.integral-calculator.com/, gives an answer of:

683120407264925/914039610015744 = apx. 0.747364118337452

Remember, that is the probability of any face appearing six times. We were asked to find the probability of the alternative, any face appearing at least six times first. To find that, subtract the above from one:

1 - 0.747364118337452 = 0.252635881662548

My thanks to Ace2 for the problem!